Moment bounds for IID sequences under sublinear expectations*

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Abstract

In this paper, with the notion of independent identically distributed (IID) random variables under sublinear expectations introduced by Peng [7-9], we investigate moment bounds for IID sequences under sublinear expectations. We can obtain a moment inequality for a sequence of IID random variables under sublinear expectations. As an application of this inequality, we get the following result: For any continuous function φ satisfying the growth condition $|\varphi(x)| \leq C(1+|x|^p)$ for some $C > 0$, $p \geq 1$ depending on φ, central limit theorem under sublinear expectations obtained by Peng [8] still holds.

Keywords moment bound, sublinear expectation, IID random variables, G-normal distribution, central limit theorem.
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1 Introduction

In classical probability theory, it is well known that for IID random variables with $E[X_1] = 0$ and $E[|X_1|^r] < \infty$ $(r \geq 2)$, $E[|S_n|^r] = O(n^{\frac{r}{2}})$ holds, and hence

$$\sup_{m\geq 0} E[|S_{m+n} - S_m|^r] = O(n^{\frac{r}{2}}). \tag{1}$$

Bounds of this kind are potentially useful to obtain limit theorems, especially strong laws of large numbers, central limit theorems and laws of the iterated logarithm (see, for example, Serfling [10] and Stout [11], Chapter 3.7).

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Since the paper (Artzner et al. [1]) on coherent risk measures, people are more and more interested in sublinear expectations (or more generally, convex expectations, see Föllmer and Schied [4] and Frittelli and Rossaza Gianin [5]). By Peng [9], we know that a sublinear expectation $\hat{E}$ can be represented as the upper expectation of a subset of linear expectations $\{E_\theta : \theta \in \Theta\}$, i.e., $\hat{E}[\cdot] = \sup_{\theta\in\Theta} E_\theta[\cdot]$. In most cases, this subset is often treated as an uncertain model of probabilities $\{P_\theta : \theta \in \Theta\}$ and the notion of sublinear expectation provides a robust way to measure a risk loss X. In fact, nonlinear expectation theory provides many rich, flexible and elegant tools.

In this paper, we are interested in

$$\overline{E}[\cdot] = \sup_{P\in\mathcal{P}} E_P[\cdot],$$

where $\mathcal{P}$ is a set of probability measures. The main aim of this paper is to obtain moment bounds for IID sequences under sublinear expectations.

This paper is organized as follows: in section 2, we give some notions and lemmas that are useful in this paper. In section 3, we give our main results including the proofs.

2 Preliminaries

In this section, we introduce some basic notions and lemmas. For a given set $\mathcal{P}$ of multiple prior probability measures on $(\Omega, \mathcal{F})$, let $\mathcal{H}$ be the set of random variables on $(\Omega, \mathcal{F})$.

For any $\xi \in \mathcal{H}$, we define a pair of so-called maximum-minimum expectations $(\overline{E}, \underline{E})$ by

$$\overline{E}[\xi] := \sup_{P\in\mathcal{P}} E_P[\xi], \quad \underline{E}[\xi] := \inf_{P\in\mathcal{P}} E_P[\xi].$$

Without confusion, here and in the sequel, $E_P[\cdot]$ denotes the classical expectation under probability measure P.

Obviously, $\overline{E}$ is a sublinear expectation in the sense that

Definition 2.1 (see Peng [8, 9]). Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We assume that all constants are in $\mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of our "random variables". A nonlinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \mapsto R$ satisfying the following properties: for all X, $Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}[X] \geq \hat{E}[Y]$.

(b) Constant preserving: $\hat{E}[c] = c$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a nonlinear expectation space (compare with a probability space $(\Omega, \mathcal{F}, P)$). We are mainly concerned with sublinear expectation where the expectation $\hat{E}$ satisfies also

(c) Sub-additivity: $\hat{E}[X] - \hat{E}[Y] \leq \hat{E}[X - Y]$.

(d)Positive homogeneity: $\hat{E}[\lambda X] = \lambda\hat{E}[X]$, $\forall\lambda \geq 0$.

If only (c) and (d) are satisfied, $\hat{E}$ is called a sublinear functional.

The following representation theorem for sublinear expectations is very useful (see Peng [9] for the proof).

Lemma 2.1. Let $\hat{E}$ be a sublinear functional defined on $(\Omega, \mathcal{H})$, i.e., (c) and (d) hold for $\hat{E}$. Then there exists a family $\{E_\theta : \theta \in \Theta\}$ of linear functionals on $(\Omega, \mathcal{H})$ such that

$$\hat{E}[X] = \max_{\theta\in\Theta} E_\theta[X]. \tag{2}$$

If (a) and (b) also hold, then E_θ are linear expectations for $\theta \in \Theta$. If we make furthermore the following assumption: (H) For each sequence $\{X_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that $X_n(\omega) \downarrow 0$ for ω, we have $\hat{E}[X_n] \downarrow 0$. Then for each $\theta \in \Theta$, there exists a unique (σ-additive) probability measure P_θ defined on $(\Omega, \sigma(\mathcal{H}))$ such that

$$E_\theta[X] = \int_\Omega X(\omega)\mathrm{d}P_\theta(\omega), \quad X \in \mathcal{H}. \tag{3}$$

Remark 2.1. Lemma 2.1 shows that in most cases, a sublinear expectation indeed is a supremum expectation. That is, if $\hat{E}$ is a sublinear expectation on $\mathcal{H}$ satisfying (H), then there exists a set (say $\hat{\mathcal{P}}$) of probability measures such that

$$\hat{E}[\xi] = \sup_{P\in\hat{\mathcal{P}}} E_P[\xi], \quad -\hat{E}[-\xi] = \inf_{P\in\hat{\mathcal{P}}} E_P[\xi].$$

Therefore, without confusion, we sometimes call supremum expectations as sublinear expectations.

Moreover, a supremum expectation $\overline{E}$ can generate a pair (V, v) of capacities denoted by

$$V(A) := \overline{E}[I_A], \quad v(A) := -\overline{E}[-I_A], \quad \forall A \in \mathcal{F}.$$

It is easy to check that the pair of capacities satisfies

$$V(A) + v(A^c) = 1, \quad \forall A \in \mathcal{F}$$

where A^c is the complement set of A.

The following is the notion of IID random variables under sublinear expectations introduced by Peng [7-9].

Definition 2.2 (**IID under sublinear expectations**). **Independence:** Suppose that $Y_1, Y_2, \cdots, Y_n$ is a sequence of random variables such that $Y_i \in \mathcal{H}$. Random variable Y_n is said to be independent of $X := (Y_1, \cdots, Y_{n-1})$ under $\overline{E}$, if for each measurable function φ on R^n with $\varphi(X, Y_n) \in \mathcal{H}$ and $\varphi(x, Y_n) \in \mathcal{H}$ for each $x \in R^{n-1}$, we have

$$\overline{E}[\varphi(X, Y_n)] = \overline{E}[\overline{\varphi}(X)],$$

where $\overline{\varphi}(x) := \overline{E}[\varphi(x, Y_n)]$ and $\overline{\varphi}(X) \in \mathcal{H}$.

Identical distribution: Random variables X and Y are said to be identically distributed, denoted by $X \sim Y$, if for each measurable function φ such that $\varphi(X),\ \varphi(Y) \in \mathcal{H}$,

$$\overline{E}[\varphi(X)] = \overline{E}[\varphi(Y)].$$

IID random variables: A sequence of random variables $\{X_i\}_{i=1}^{\infty}$ is said to be IID, if $X_i \sim X_1$ and X_{i+1} is independent of $Y := (X_1, \cdots, X_i)$ for each $i \geq 1$.

Definition 2.3 (**Pairwise independence**, see Marinacci [6]). Random variable X is said to be pairwise independent of Y under capacity $\hat{V}$, if for all subsets D and $G \in \mathcal{B}(R)$,

$$\hat{V}(X \in D, Y \in G) = \hat{V}(X \in D)\hat{V}(Y \in G).$$

The following lemma shows the relation between Peng's independence and pairwise independence.

Lemma 2.2. Suppose that $X, Y \in \mathcal{H}$ are two random variables. $\overline{E}$ is a sublinear expectation and (V, v) is the pair of capacities generated by $\overline{E}$. If random variable X is independent of Y under $\overline{E}$, then X also is pairwise independent of Y under capacities V and v.

Proof. If we choose $\varphi(x, y) = I_D(x)I_G(y)$ for $\overline{E}$, by the definition of Peng's independence, it is easy to obtain

$$V(X \in D, Y \in G) = V(X \in D)V(Y \in G).$$

Similarly, if we choose $\varphi(x, y) = -I_D(x)I_G(y)$ for $\overline{E}$, it is easy to obtain

$$v(X \in D, Y \in G) = v(X \in D)v(Y \in G).$$

The proof is complete.

Let $C_b(R^n)$ denote the space of bounded and continuous functions, let $C_{l,Lip}(R^n)$ denote the space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \quad \forall x, y \in R^n,$$

for some $C > 0$, $m \in N$ depending on φ and let $C_{b,Lip}(R^n)$ denote the space of bounded functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C|x - y| \quad \forall x, y \in R^n,$$

for some $C > 0$ depending on φ.

From now on, we consider the following sublinear expectation space $(\Omega, \mathcal{H}, \overline{E})$: if $X_1, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(R^n)$.

Definition 2.4 (***G*-normal distribution**, see Definition 10 in Peng [7]). A random variable $\xi \in \mathcal{H}$ under sublinear expectation $\widetilde{E}$ with $\overline{\sigma}^2 = \widetilde{E}[\xi^2]$, $\underline{\sigma}^2 = -\widetilde{E}[-\xi^2]$ is called G-normal distribution, denoted by $\mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, if for any function $\varphi \in C_{l,Lip}(R)$, write $u(t, x) := \widetilde{E}[\varphi(x + \sqrt{t}\xi)]$, $(t, x) \in [0, \infty) \times R$, then u is the unique viscosity solution of PDE:

$$\partial_t u - G(\partial^2_{xx} u) = 0, \quad u(0, x) = \varphi(x),$$

where $G(x) := \frac{1}{2}(\overline{\sigma}^2 x^+ - \underline{\sigma}^2 x^-)$ and $x^+ := \max\{x, 0\}$, $x^- := (-x)^+$.

With the notion of IID under sublinear expectations, Peng shows central limit theorem under sublinear expectations (see Theorem 5.1 in Peng [8]).

Lemma 2.3 (**Central limit theorem under sublinear expectations**). Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of IID random variables. We further assume that $\overline{E}[X_1] = \overline{E}[-X_1] = 0$. Then the sequence $\{\overline{S}_n\}_{n=1}^{\infty}$ defined by $\overline{S}_n := \frac{1}{\sqrt{n}} \sum\limits_{i=1}^{n} X_i$ converges in law to ξ, i.e.,

$$\lim_{n\to\infty} \overline{E}[\varphi(\overline{S}_n)] = \widetilde{E}[\varphi(\xi)],$$

for any continuous function φ satisfying linear growth condition (i.e., $|\varphi(x)| \le C(1+|x|)$ for some $C > 0$ depending on φ), where ξ is a G-normal distribution.

3 Main results and proofs

Theorem 3.1. Let a random sequence $\{X_n\}_{n=1}^{\infty}$ be IID under $\overline{E}$. Denote $S_n := \sum\limits_{i=1}^{n} X_i$. Assume that $\overline{E}[X_1] = \overline{E}[-X_1] = 0$. Then for each $r > 2$, there exists a positive constant K_r not depending on n such that for all $n \in N$,

$$\sup_{m\ge 0} \overline{E}[|S_{m+n} - S_m|^r] \le K_r n^{\frac{r}{2}}.$$

Proof. Let $r = \theta + \gamma$, where $\theta \in N, \theta \ge 2$ and $\gamma \in (0, 1]$. For simplicity, write

$$S_{m,n} := S_{m+n} - S_m,$$

$$a_n := \sup_{m\ge 0} \overline{E}[|S_{m,n}|^r].$$

Firstly, we shall show that there exists a positive constant C_r not depending on n such that for all $n \in N$,

$$\overline{E}[|S_{m,2n}|^r] \le 2a_n + C_r a_n^{1-\gamma} n^{\frac{\gamma r}{2}}. \tag{4}$$

In order to prove (4), we show the following inequalities for all $n \in N$:

$$\overline{E}[|S_{m,2n}|^r] \le 2a_n + 2^{\theta+1}(\overline{E}[|S_{m,n}|^{\gamma}|S_{m+n,n}|^{\theta}] + \overline{E}[|S_{m,n}|^{\theta}|S_{m+n,n}|^{\gamma}]), \tag{5}$$

$$\overline{E}[|S_{m,n}|^{\gamma}|S_{m+n,n}|^{\theta}] \le a_n^{1-\gamma}(\overline{E}[|S_{m,n}||S_{m+n,n}|^{\theta-1+\gamma}])^{\gamma}, \tag{6}$$

$$\overline{E}[|S_{m,n}|^{\theta}|S_{m+n,n}|^{\gamma}] \le a_n^{1-\gamma}(\overline{E}[|S_{m,n}|^{\theta-1+\gamma}|S_{m+n,n}|])^{\gamma}, \tag{$6'$}$$

$$\overline{E}[|S_{m,n}||S_{m+n,n}|^{\theta-1+\gamma}] \le D_r n^{\frac{r}{2}}, \tag{7}$$

$$\overline{E}[|S_{m,n}|^{\theta-1+\gamma}|S_{m+n,n}|] \le D_r n^{\frac{r}{2}}, \tag{$7'$}$$

where D_r is a positive constant not depending on n.

To prove (5). Elementary estimates yield the following inequality (*):

$$\begin{aligned}
&|S_{m,2n}|^r = |S_{m,n} + S_{m+n,n}|^{\theta+\gamma} \le (|S_{m,n}| + |S_{m+n,n}|)^{\theta}(|S_{m,n}| + |S_{m+n,n}|)^{\gamma} \\
\le\ & \textstyle\sum_{i=0}^{\theta} C_\theta^i |S_{m,n}|^{\theta-i}|S_{m+n,n}|^i(|S_{m,n}|^{\gamma} + |S_{m+n,n}|^{\gamma}) \\
\le\ & |S_{m,n}|^{\theta+\gamma} + |S_{m+n,n}|^{\theta+\gamma} + 2\textstyle\sum_{i=0}^{\theta} C_\theta^i(|S_{m,n}|^{\gamma}|S_{m+n,n}|^{\theta} + |S_{m,n}|^{\theta}|S_{m+n,n}|^{\gamma}) \\
\le\ & |S_{m,n}|^{\theta+\gamma} + |S_{m+n,n}|^{\theta+\gamma} + 2^{\theta+1}(|S_{m,n}|^{\gamma}|S_{m+n,n}|^{\theta} + |S_{m,n}|^{\theta}|S_{m+n,n}|^{\gamma}).
\end{aligned}$$

Since $\{X_n\}_{n=1}^{\infty}$ is a IID random sequence, by the definition of IID under sublinear expectations,

$$a_n = \sup_{m\geq 0}\overline{E}[|S_{m,n}|^r] = \sup_{m\geq 0}\overline{E}[|S_{m+n,n}|^r].$$

Taking $\overline{E}[\cdot]$ on both sides of (*), we have

$$\overline{E}[|S_{m,2n}|^r] \leq 2a_n + 2^{\theta+1}(\overline{E}[|S_{m,n}|^{\gamma}|S_{m+n,n}|^{\theta}] + \overline{E}[|S_{m,n}|^{\theta}|S_{m+n,n}|^{\gamma}]).$$

Hence, (5) holds.

Since the proof of $(6')$ is very similar to that of (6), we only prove (6). Without loss of generality, we assume $\gamma \in (0,1)$. By Hölder's inequality,

$$\begin{aligned}\overline{E}[|S_{m,n}|^{\gamma}|S_{m+n,n}|^{\theta}] &\leq (\overline{E}[|S_{m,n}||S_{m+n,n}|^{\theta-1+\gamma}])^{\gamma}(\overline{E}[|S_{m+n,n}|^{\frac{\theta-\gamma(\theta-1+\gamma)}{1-\gamma}}])^{1-\gamma}\\ &\leq a_n^{1-\gamma}(\overline{E}[|S_{m,n}||S_{m+n,n}|^{\theta-1+\gamma}])^{\gamma}.\end{aligned}$$

This proves (6).

To prove (7). By the definition of IID under sublinear expectations and Schwarz's inequality, we have

$$\overline{E}[|S_{m,n}||S_{m+n,n}|^{\theta-1+\gamma}] = \overline{E}[|S_{m,n}|]\overline{E}[|S_{m+n,n}|^{\theta-1+\gamma}] \leq (\overline{E}[|S_{m,n}|^2])^{\frac{1}{2}}\overline{E}[|S_{m+n,n}|^{\theta-1+\gamma}]. \quad (8)$$

Next we prove

$$\overline{E}[S_{m,n}^2] \leq n\overline{E}[X_1^2], \quad \forall m \geq 0.$$

Indeed, using the definition of IID under sublinear expectations again, we have

$$\begin{aligned}&\overline{E}[S_{m,n}^2] = \overline{E}[(S_{m,n-1}+X_{m+n})^2] = \overline{E}[S_{m,n-1}^2 + 2S_{m,n-1}X_{m+n} + X_{m+n}^2]\\ \leq\ &\overline{E}[S_{m,n-1}^2] + \overline{E}[X_{m+n}^2] \leq \cdots = n\overline{E}[X_1^2].\end{aligned}$$

So

$$\overline{E}[S_{m,n}^2] \leq n\overline{E}[X_1^2] \quad (9)$$

and

$$\overline{E}[S_{m+n,n}^2] \leq n\overline{E}[X_1^2] \quad (10)$$

hold. On the other hand, by Hölder's inequality,

$$\overline{E}[|S_{m+n,n}|^{1+\gamma}] \leq (\overline{E}[S_{m+n,n}^2])^{\frac{1+\gamma}{2}} \leq n^{\frac{1+\gamma}{2}}(\overline{E}[X_1^2])^{\frac{1+\gamma}{2}}. \quad (11)$$

If $\theta = 2$, (7) follows from (8), (9), (10) and (11). If $\theta > 2$, we inductively assume

$$\overline{E}[|S_{m+n,n}|^{\theta-1+\gamma}] \leq M_r n^{\frac{\theta-1+\gamma}{2}}, \quad (12)$$

where M_r is a positive constant not depending on n. Then (8), (9) and (12) yield (7). In a similar manner, we can prove that $(7')$ holds.

From (5)-$(7')$, it is easy to check that (4) holds. From (4), we can obtain that for all $n \in N$,

$$a_{2n} \leq 2a_n + C_r a_n^{1-\gamma} n^{\frac{\gamma r}{2}}.$$

By induction, there exists a positive constant C_r' not depending on n such that $a_n \le C_r' n^{\frac{r}{2}}$ for all $n \in \{2^k : k \in N \bigcup \{0\}\}$.

If n is any positive integer, it can be written in the form

$$n = 2^k + v_1 2^{k-1} + \cdots + v_k \le 2^k + 2^{k-1} + \cdots + 1$$

where $2^k \le n < 2^{k+1}$ and each v_j is either 0 or 1. Then $S_{m,n}$ can be written as the sum of $k+1$ groups of sums containing $2^k, v_1 2^{k-1}, \cdots$ terms and using Minkowski's inequality,

$$\begin{aligned} a_n &\le \sup_{m\ge 0}[(\overline{E}[|S_{m+v_k+\cdots+v_1 2^{k-1}, 2^k}|^r])^{\frac{1}{r}} + \cdots + (\overline{E}[|S_{m,v_k}|^r])^{\frac{1}{r}}]^r \\ &\le \; C_r'[2^{\frac{k}{2}} + \cdots + 1]^r = C_r'[\frac{2^{\frac{k+1}{2}}-1}{2^{\frac{1}{2}}-1}]^r \le K_r n^{\frac{r}{2}}. \end{aligned}$$

The proof is complete.

Remark 3.1. (i) From the proof of Theorem 3.1, we can check that the assumption of IID under $\overline{E}$ can be replaced by the weaker assumption that $\{X_n\}_{n=1}^{\infty}$ is a IID random sequence under $\overline{E}$ with respect to the following functions

$$\varphi_1(x) = x; \quad \varphi_2(x) = -x;$$

$$\varphi_3(x_1, \cdots, x_n) = |x_1 + \cdots + x_n|^r, \quad n = 1, 2, \cdots, \quad r \ge 2;$$

$$\varphi_4(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = |x_1 + \cdots + x_m||x_{m+1} + \cdots + x_{m+n}|^p, m, n = 1, 2, \cdots, \quad p > 1;$$

and

$$\varphi_5(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = |x_1 + \cdots + x_m|^p|x_{m+1} + \cdots + x_{m+n}|, m, n = 1, 2, \cdots, \quad p > 1.$$

(ii) A close inspection of the proof of Theorem 3.1 reveals that the definition of IID under sublinear expectations plays an important role in the proof. The proof of Theorem 3.1 is very similar to the classical arguments, e.g., in Theorem 1 of Birkel [2].

Applying Theorem 3.1, we can obtain the following result:

Theorem 3.2. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of IID random variables. We further assume that $\overline{E}[X_1] = \overline{E}[-X_1] = 0$. Then the sequence $\{\overline{S}_n\}_{n=1}^{\infty}$ defined by $\overline{S}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ converges in law to ξ, i.e.,

$$\lim_{n\to\infty} \overline{E}[\varphi(\overline{S}_n)] = \widetilde{E}[\varphi(\xi)], \tag{13}$$

for any continuous function φ satisfying the growth condition $|\varphi(x)| \le C(1+|x|^p)$ for some $C > 0$, $p \ge 1$ depending on φ, where ξ is a G-normal distribution.

Proof. Indeed, we only need to prove that (13) holds for the $p > 1$ cases. Let φ be an arbitrary continuous function with growth condition $|\varphi(x)| \le C(1+|x|^p)$ ($p > 1$). For each $N > 0$, we can find two continuous functions φ_1, φ_2 such that $\varphi = \varphi_1 + \varphi_2$, where φ_1 has a compact support and $\varphi_2(x) = 0$ for $|x| \le N$, and $|\varphi_2(x)| \le |\varphi(x)|$ for all x. It is clear that $\varphi_1 \in C_b(R)$ and

$$|\varphi_2(x)| \le \frac{2C(1+|x|^{p+1})}{N}, \quad \text{for } \; x \in R.$$

Thus

$$\begin{aligned} |\overline{E}[\varphi(\overline{S}_n)] - \widetilde{E}[\varphi(\xi)]| &= |\overline{E}[\varphi_1(\overline{S}_n) + \varphi_2(\overline{S}_n)] - \widetilde{E}[\varphi_1(\xi) + \varphi_2(\xi)]| \\ &\leq |\overline{E}[\varphi_1(\overline{S}_n)] - \widetilde{E}[\varphi_1(\xi)]| + |\overline{E}[\varphi_2(\overline{S}_n)] - \widetilde{E}[\varphi_2(\xi)]| \\ &\leq |\overline{E}[\varphi_1(\overline{S}_n)] - \widetilde{E}[\varphi_1(\xi)]| + \tfrac{2C}{N}(2 + \overline{E}[|\overline{S}_n|^{p+1}] + \widetilde{E}[|\xi|^{p+1}]). \end{aligned}$$

Applying Theorem 3.1, we have $\sup\limits_n \overline{E}[|\overline{S}_n|^{p+1}] < \infty$. So the above inequality can be rewritten as

$$|\overline{E}[\varphi(\overline{S}_n)] - \widetilde{E}[\varphi(\xi)]| \leq |\overline{E}[\varphi_1(\overline{S}_n)] - \widetilde{E}[\varphi_1(\xi)]| + \frac{\overline{C}}{N},$$

where $\overline{C} = 2C(2 + \sup\limits_n \overline{E}[|\overline{S}_n|^{p+1}] + \widetilde{E}[|\xi|^{p+1}])$. From Lemma 2.3, we know that (13) holds for any $\varphi \in C_b(R)$ with a compact support. Thus, we have $\limsup\limits_{n\to\infty} |\overline{E}[\varphi(\overline{S}_n)] - \widetilde{E}[\varphi(\xi)]| \leq \frac{\overline{C}}{N}$. Since N can be arbitrarily large, $\overline{E}[\varphi(\overline{S}_n)]$ must converge to $\widetilde{E}[\varphi(\xi)]$. The proof of Theorem 3.2 is complete.

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